POISSON COHOMOLOGY: OLD AND NEW

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Poisson geometry is a vast generalization of the Hamiltonian setting of classical mechanics, described abstractly by symplectic geometry. Its applications in theoretical physics have become clearer during recent decades, where the deformation theory of Poisson structures intertwines with deformation quantization.

The deformation of Poisson structures is also related to the theory of normal forms in Poisson geometry and, consequently, to the problem of classifications of normal forms. The deformation theory of a Poisson manifold (M, Π) is governed by the cohomology of the Lichnerowicz complex $(\mathfrak{X}^\bullet, \mathrm{d}_\Pi)$. The cohomology groups are denoted $H^\bullet_\Pi(M)$ and coined Poisson cohomology groups in the literature.

For an important class of Poisson structures, the singularities can be encoded in a suitable vector bundle E and analyzed by means of geometric techniques. More specifically, such Poisson structures Π are obtained from a symplectic Lie algebroid (E, ρ, ω), that is, a vector bundle $E \to M$ with an anchor $\rho\colon E \to \mathrm{T}M$ and a closed, non-degenerate two-form $\omega \in \Omega^2(E)$. A notable instance of this definition is b-symplectic geometry, where $E = {}^b\mathrm{T}M$ is called the b-tangent bundle. Sections of ${}^b\mathrm{T}M$ can be identified with smooth vector fields $X \in \mathfrak{X}(M)$ which are tangent to an embedded hypersurface $Z \subset M$. A generalization of this setting is b^m-symplectic geometry, where vector fields are assumed to be tangent to Z at least of order m.

The systematic use of these techniques in the study of Poisson cohomology was pioneered by Guillemin, Miranda and Pires in the case of b-geometry. The complex of sections of ${}^b\mathrm{T}M$ is naturally included in the complex of smooth multi-vector fields and admits a restriction of the operator d_Π. Consequently, it is a sub-complex of the Lichnerowicz complex. A comparison lemma due to Mărcuţ and Osorno shows this inclusion morphism induces an isomorphism at the level of cohomology.

In this master thesis we expand on these techniques to answer the following question posed by Alan Weinstein: is the inclusion morphism an isomorphism in b^m-symplectic geometry? We show the quotient complex $\mathfrak{X}^\bullet_Q(M)$ obtained from the short exact sequence of complexes

$$0 \longrightarrow {}^{b^m}\mathfrak{X}^\bullet(M) \xrightarrow{\ i\ } \mathfrak{X}^\bullet(M) \xrightarrow{\ \pi\ } \mathfrak{X}^\bullet_Q(M) \longrightarrow 0,$$

which measures the obstruction to i being an isomorphism in cohomology, has infinite-dimensional cohomology groups. By showing this complex can be localized to the degeneracy locus $Z \subseteq M$ of Π, we are able to use normal form theory to completely describe the cohomology groups of $\mathfrak{X}^\bullet_Q(M)$ and, using a long exact sequence, to ultimately recover the Poisson cohomology of a general b^m-Poisson manifold. We have been able to give an interpretation of these infinite contributions by appropriately deblogging the singularity of the Poisson structure and analyzing the asymptotic behavior of the coboundary terms in the cohomology equations.