Matlis duality, inverse systems and classification of Artin algebras

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The objective of my master's final project is to study notions about injective modules, Matlis duality and Macaulay's correspondence in order to show how to use Macaulay's duality for classifying Artin algebras.

Firstly, let's consider R a commutative ring and E an R-module. We say that E is *injective* if and only if, for all injective morphism $i\colon A \to B$ and for all morphism $f\colon A \to E$, where A and B are R- modules, a morphism $g\colon B \to E$ exists such that the following diagram commute. With that definition, one is able to prove the existence of injective hulls of modules. From this result, we can deduce the existence of minimal injective resolutions of R-modules.

When R is a Noetherian ring, there are different results about injective modules. In this case, we can define what a Bass number is and study the relation between this numbers and the minimal injective resolution of a finite R- module.

All this concepts are necessary to talk about Matlis duality which ensures an isomorphism between Artin and Noetherian modules. Given A a R-module, let $(R, \mathfrak{a}, \mathbf{k})$ Noetherian local ring, its dual will be $A^\vee = \mathrm{Hom}_R(A, E)$, E an injective hull of $\mathbf{k}$, by Matlis duality. It was written and proved by Eben Matlis at 1958.

Macaulay's correspondence is the particular case of Matlis duality when

$$R = \mathbf{k}[\![x_1, \dots, x_n]\!],$$

the ring of formal series, with maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. This correspondence is useful to define Hilbert function in a simplified way.

All these previous theorems and definitions are necessary in order to study study Gorenstein rings, level and compressed algebras. We focus on the case of Gorenstein rings and we study the relation between them and Artin rings. Also, one can define Irrabino's Q-decomposition of the associated graded ring of an Artinian s-level local k-algebra when we study level rings. One of the main important results of the project is that an isomorphism between two Artinian s-level algebras is defined by a matrix using Macaulay's inverse system.

In order to make more tangible some of the previous results, we do some computations with Singular and using INVERSE-SYST.LIB by J. Elias.

With Singular, one can prove the existence of an isomorphism between some models for A and its inverse system when A is an Artin Gorenstein local $\mathbf{k}$-algebra with Hilbert function $\mathrm{HF}_A = \{1, 3, 3, 1\}$.