Convergence to the Brownian Motion

Marc Cano i Cànovas

Extended Abstract

During the 19th century, scientists started developing the discipline of statistical mechanics. Basically, they started treating physical systems mathematically. In 1859, James Clerk Maxwell presented a work on the kinetic theory of gases where he assumed that the gas particles move in random directions at random velocities. This was the starting point for the development of the statistical physics during the second half of the 19th century. During this period of time, Thorvald N. Thiele, in 1880, published a paper where he described the mathematics behind the *Brownian motion*.

The Brownian motion describes the random movement of particles suspended in a liquid or a gas. Despite it was first described by the botanist Robert Brown, in 1827, while he was looking at pollen particles through a microscope, it was not until the end of the 19th century that the mathematics behind this motion were addressed. Furthermore, it was not until 1900 when Louis Bachelier modeled for the first time, and under the supervision of Henri Poincaré, the stochastic process that we know as the Brownian motion.

This Brownian motion is the protagonist of this thesis. What we do is prove different results regarding some type of convergence towards this Brownian motion.

One of the results that we see is a classical result which is the Donsker's theorem. For this result we follow the first chapter, and part of the second, of the book *Convergence of Probability Measures* by Patrick Billingsley. To reinforce this two chapters we also follow the book *Curs de Probabilitats* by David Nualart and Marta Sanz.

In order to state and prove this result we first discuss about different notions of convergence such as the weak convergence, the convergence in distribution or the convergence in probability. Also, we define the concept of tightness. Moreover, we discuss this concept of tightness, and also the notion of weak convergence, in the set of continuous functions on $[0, 1]$. To finish, we see the definitions of the Wiener measure and the Brownian motion.

With all this previous work, we are able to state the Donsker's theorem and prove that the stochastic processes that it defines converge towards the Brownian motion in distribution.

Another result that we prove is the convergence in distribution of a particular type of stochastic processes. This processes that we define were presented by Mark Kac but was Daniel Stroock who explicitly proved their convergence. This result can be found in the work of Stroock, *Topics in Stochastic Differential Equations*. Even so, we follow a presentation made by Xavier Bardina on December 18, 2014 at Bucuresti called *On the Kac-Stroock Approximations* in order to achieve the proof of this result.

The last result that we see is the almost sure convergence of the uniform transport processes towards the Brownian motion. In order to prove this result we follow the paper written by Richard J. Griego, David Heath and Alberto Ruiz-Moncayo, *Almost Sure Convergence of Uniform Transport Processes to Brownian Motion*.

We also use classical books as *An Introduction to Probability Theory and its Applications* by William Feller or *Studies in the Theory of Random Processes* by Anatoliy V. Skorokhod to complement the proof.

Furthermore, we talk about a couple of results that extend the result of almost sure convergence. To do so, we use some results that we can find in the paper *Rate of Convergence of Uniform Transport Processes to Brownian Motion and Application to Stochastic Integrals* by Luis G. Gorostiza and Richard J. Griego and in the paper *On the Convergence of Ordinary Integrals to Stochastic Integrals* by Eugene Wong and Moshe Zakai.